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A class of orthogonal polynomials suggested by a trigonometric Hamiltonian: Symmetric states

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A new subclass of the Jacobi polynomials arising in the exact analytical solution of the one-dimensional Schrödinger equation with a trigonometric potential has been introduced. The polynomials which consist of a free parameter are not ultraspherical polynomials and have been simply named the $\mathcal{T}$-polynomials since they are generated by a trigonometric Hamiltonian. In certain sense, it is shown that the $\mathcal{T}$-polynomials can be regarded as a generalisation of the airfoil polynomials or the Chebyshev polynomials of the third kind. This paper is intended to discuss the basic properties of the polynomials so defined.

KEY WORDS: Schrödinger equation, Exactly solvable hamiltonians, special functions, classical orthogonal polynomials

AMS subject classification: 33C45, 81Q05, 33C05

1. Introduction

The classical orthogonal polynomials (COPs) associated with the names of Hermite, Laguerre and Jacobi play a starring role in theoretical physics, applied mathematics, numerical analysis and scientific computing. A sequence of real polynomials $p_n(x)$ is orthogonal over (a, b) relative to a non-negative weighting function $\rho(x)$ if

$$\langle p_m, p_n\rangle_\rho := \int_a^b \rho(x) p_m(x) p_n(x) \mathrm{d}x = \mathcal{N}_n^2 \delta_{mn}, \quad n \in \mathbb{N}, \tag{1}$$

where $\mathcal{N}_n$ is the norm of $p_n(x)$, δ_{mn} the Kronecker's delta symbol, and $\mathbb{N}$ denotes the set of non-negative integers $n = 0, 1, \ldots$ The COPs satisfy second order differential equations, and also have the striking feature that their derivatives form again orthogonal systems. There are several additional properties common to all COPs, however, an extensive review of orthogonal polynomials is outside the scope of this work, and an excellent survey can be found in [1].

It is known that the two parameters family of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ for α and β both greater than -1, the one parameter family of Laguerre polynomials $L_n^{(\alpha)}(x)$ for $\alpha > -1$ and Hermite polynomials $H_n(x)$ are the only polynomial sets, which possess orthogonality in the sense of (1) over the real x-intervals $(-1, 1)$, $(0, \infty)$, and $(-\infty, \infty)$, respectively, relative to certain weight functions $\rho(x)$. It is important to note that the COPs occur naturally in many branches of physical sciences. Especially, there are many problems in quantum mechanics leading to the COPs. For instance, Hermite polynomials arise in the solution of the harmonic oscillator, which has applications to various types of oscillations in crystals and molecules. As another typical example, the treatment of the Schrödinger equation for a particle in a central force field, such as the hydrogen atom, can be accomplished by means of the Laguerre polynomials.

On the other hand, the Jacobi polynomials appear in many cases with equal integral parameters so that $\alpha = \beta = m$, where $m \in \mathbb{N}$. These polynomials are faced, for example, in the study of Laplace's equation in spherical coordinates and are closely related to the spherical harmonics. The simplest case in which $m = 0$ yields the famous Legendre polynomials $P_n(x)$. More generally, the Jacobi polynomials having two parameters equal to a number which is not necessarily an integer, $\alpha = \beta = \lambda - 1/2$ say, where $\lambda \in \mathbb{R}$, are frequently encountered in practice too, and known as the *ultraspherical* or Gegenbauer polynomials denoted by $C_n^{(\lambda)}(x)$. Two celebrated examples are the Chebyshev polynomials of the first and second kind, which are ultraspherical polynomials with $\lambda = 0$ and 1, respectively.

In recent articles, Marmorino [2] and Taşeli [3] dealt with the exact solutions of squared cotangent and tangent potentials. In fact, if we recall the simple trigonometric identities $\tan^2 x = \sec^2 x - 1$ and $\cot^2 x = \csc^2 x - 1$ these potentials could be regarded as special cases of the Pöschl–Teller potential hole [4]. The main goal of this article is to examine the orthogonal polynomials appearing in the analytical solutions of the associated eigenvalue problems. In section 2, the one-dimensional Schrödinger operator with such a trigonometric potential and its explicit eigensolutions in closed form are presented. The complete orthonormal set of the symmetric state wavefunctions which consists of non-ultraspherical Jacobi polynomials is then introduced. These polynomials are examined in section 3 under the title $\mathcal{T}$-polynomials of the first kind. The last section is devoted to a discussion of the limiting case of the potential parameter and concluding remarks.

2. The Hamiltonian with a trigonometric potential

Let us consider the one-dimensional Schrödinger equation $H\Psi = E\Psi$ over $\theta \in (-\pi, \pi)$, with a trigonometric Hamiltonian

$$
\begin{aligned}
H(\theta; \mu) &= -\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + v(\theta; \mu), \\
v(\theta; \mu) &= \frac{1}{4}\mu(\mu+1)\sec^2 \frac{1}{2}\theta = \frac{1}{2}\mu(\mu+1)(1+\cos\theta)^{-1}
\end{aligned}
\tag{2}
$$

in which $\mu(\mu+1) > 0$ for a well potential. Thus the potential parameter μ can be a real number either greater than 0 or less than -1, that is, $\mu \notin (-1, 0)$. However, the differential operator has an obvious scaling relationship

$$H(\theta; \mu) = H(\theta; -\mu - 1) \tag{3}$$

and, therefore, we may take $\mu > 0$ without any loss of generality.

The mathematical problem in question is a singular Sturm–Liouville system because of the unboundedness of the trigonometric potential at $\pm\pi$, which implies that the wavefunction $\Psi(\theta; \mu)$ must vanish at the boundaries. Clearly, such a wavefunction will be square integrable over the θ-domain. Furthermore, as a consequence of the reflection symmetry of the Hamiltonian under the replacement of θ by $-\theta$, i.e.,

$$H(\theta; \mu) = H(-\theta; \mu), \quad \theta \in (-\pi, \pi) \;\; \mu > 0 \tag{4}$$

the eigenfunctions are either even or odd functions of θ corresponding to the symmetric and antisymmetric energy levels. Note that since $v(\theta; \mu) \geqslant \mu(\mu+1)/4 > 0$ for all θ and μ, only positive values of $E(\mu)$ are admissible as the energy eigenvalues. Note also that, in this article, we shall deal with the symmetric state eigensolutions of the problem.

In the limiting case of the parameter $\mu \to 0^+$, the Hamiltonian reduces to the so-called particle-in-a-box model with

$$v(\theta; 0) = \begin{cases} 0 & \text{for } \theta \in (-\pi, \pi), \\ \infty & \text{for } |\theta| \geqslant \pi, \end{cases} \tag{5}$$

which is one of the elementary examples of an exactly solvable system. Indeed, it is standard to find out that the normalised symmetric state eigenfunctions are given by

$$\Psi_{2n}(\theta; 0) = \frac{1}{\sqrt{\pi}} \cos(n + \tfrac{1}{2})\theta = \frac{1}{\sqrt{\pi}} \cos \tfrac{1}{2}\theta \, V_n(\cos\theta) \tag{6}$$

corresponding to the eigenvalues

$$E_{2n}(0) = \frac{1}{4}(2n+1)^2 \tag{7}$$

for each $n \in \mathbb{N}$. In (6), we have set

$$V_n(x) := \cos(n + \tfrac{1}{2})\theta / \cos \tfrac{1}{2}\theta, \qquad V_0(x) = 1, \qquad V_1(x) = 2x - 1, \tag{8}$$

which is a polynomial of degree n in x for all n, when $x = \cos\theta$. This polynomial is sometimes referred to as the *airfoil polynomial* or, more appropriately, the Chebyshev polynomial of the third kind [5]. Indeed, $V_n(x)$ is directly related to the Chebyshev polynomial of the first kind. It is seen from (8) that it is defined trigonometrically parallel to the definition $T_n(x) := \cos n\theta$ of the usual first-kind Chebyshev polynomials, and satisfies a recursion which is identical in form to that of $T_n(x)$.

The structure of the exact solutions of the particle-in-a-box model suggests evidently the introduction of the non-one-to-one mapping

$$x = \cos\theta, \quad x \in (-1, 1) \tag{9}$$

for the treatment of the symmetric states, which transforms the differential equation into the form

$$\left\{(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x\frac{\mathrm{d}}{\mathrm{d}x} + E - \frac{\mu(\mu+1)}{2(1+x)}\right\}\Psi_{\mathrm{s}}(x;\mu) = 0 \tag{10}$$

subject to $\Psi_{\mathrm{s}}(-1;\mu) = 0$ for all μ, where Ψ_{s} denotes an even eigenfunction in θ. Next proposing a wavefunction of the type

$$\Psi_{\mathrm{s}}(x;\mu) = (1+x)^{(\mu+1)/2}y(x), \tag{11}$$

we avoid the use of the term proportional to $(1+x)^{-1}$ and arrive at the equation

$$(1-x^2)y'' + [\mu + 1 - (\mu+2)x]y' + \left[E - \frac{1}{4}(\mu+1)^2\right]y = 0 \tag{12}$$

for $y(x)$. However, it is more convenient to shift the argument from x to ξ

$$\xi = \frac{1}{2}(1+x), \quad \xi \in (0, 1) \tag{13}$$

and obtain the hypergeometric equation

$$\xi(1-\xi)y'' + [c - (a+b+1)\xi]y' - aby = 0 \tag{14}$$

for the transformed dependent variable $y(\xi)$. Here $c = \mu + 3/2$, and the real parameters a and b are to be taken as solutions of the non-linear system

$$a + b = \mu + 1, \quad ab = \frac{1}{4}(\mu+1)^2 - E \tag{15}$$

of two algebraic equations. This system has in fact only one significant solution which consists of

$$a = \frac{1}{2}(\mu+1) - \sqrt{E}, \qquad b = \frac{1}{2}(\mu+1) + \sqrt{E} \tag{16}$$

owing to the symmetric structure of (14) in the two parameters a and b. The solution of the hypergeometric equation, which leads to a wavefunction Ψ_{s} vanishing at the origin of the ξ-axis, is given by

$$y(\xi) = {}_2F_1(a, b; c; \xi) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{\xi^k}{k!}$$

$$= 1 + \frac{ab}{c}\frac{\xi}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{\xi^2}{2!} + \cdots \tag{17}$$

where $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$ denotes Pochhammer's symbol. Such a hypergeometric series centered at the origin is convergent if $|\xi| < 1$, i.e., in the interval $(-1, 1)$, due to the fact that the nearest singularity is located at $\xi = 1$. Nevertheless, since $c = \mu + 3/2$ is never zero or a negative integer, and $c - a - b = 1/2 > 0$ it is easy to deduce from Gauss's theorem that the series has a sum at $\xi = 1$,

$$C_1 := \lim_{\xi \to 1^-} {}_2F_1(a, b; c; \xi) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = \frac{\Gamma(c)\Gamma(\frac{1}{2})}{\Gamma(c-a)\Gamma(c-b)} \tag{18}$$

no matter what a and b are [6]. Therefore, it seems at first sight that the solution in (17) is valid on the whole physical domain $\xi \in [0, 1]$ with no restriction at all. However, the situation is completely different, and ${}_2F_1(a, b; c; \xi)$ describes indeed the required solution by a supplementary condition. The reason for this will become clear shortly.

It is a very well-known fact that the hypergeometric equation (14) possesses another pair of linearly independent solutions containing hypergeometric functions of argument $1 - \xi$. In other words, it is possible to construct series solutions about $\xi = 1$ converging in $(0, 2)$ as the origin is also a singular point of the differential equation. Therefore, for the hypergeometric function ${}_2F_1(a, b; c; \xi)$ with $c = \mu + 3/2$ and $a + b = \mu + 1$, there is an identical form

$${}_2F_1(a, b; c; \xi) = C_1\, {}_2F_1(a, b; \frac{1}{2}; 1-\xi) + C_2 g(\xi) \tag{19}$$

valid in the interval $\xi \in (0, 1)$, where

$$C_2 = \frac{\Gamma(-\frac{1}{2})\Gamma(\mu+\frac{3}{2})}{\Gamma(a)\Gamma(b)},$$
$$g(\xi) = \sqrt{1-\xi}\, {}_2F_1(\mu + \frac{3}{2} - a, \mu + \frac{3}{2} - b; \frac{3}{2}; 1-\xi). \tag{20}$$

The relation in (19) represents a typical of a larger number known as the *linear transformation formulas* which exist between the solutions of the hyprergeometric equation [7].

It is readily seen that the expression in (19) is consistent as $\xi \to 1^-$ confirming the definition in (16). However, a somewhat careful inspection shows that the function $g(\xi)$ stands out in the solution. First, the solution is no longer an even function of the original variable θ because of the factor $\sqrt{1-\xi}$ in $g(\xi)$, which contradicts the decomposition property of the set of eigenfunctions in case of a

Hamiltonian having a reflection symmetry as in (4). Moreover, it is natural to expect that each side of (19) tends to the same limit as $\xi \to 0^+$. In what follows, on utilising the functional equation [8]

$$ {}_2F_1(a, b; c; z) = (1 - z)^{c-a-b}\, {}_2F_1(c - a, c - b; c; z), \tag{21} $$

we find that the function $g(\xi)$ is expressible as

$$ g(\xi) = \xi^{-\mu-1/2}\sqrt{1-\xi}\, {}_2F_1(a - \mu, b - \mu; \frac{3}{2}; 1 - \xi) \tag{22} $$

and that

$$ g(\xi) \sim \frac{\sqrt{\pi}\,\Gamma(\mu + \frac{1}{2})}{2\Gamma(\mu + \frac{3}{2} - a)\Gamma(\mu + \frac{3}{2} - b)}\xi^{-\mu-1/2} \tag{23} $$

as $\xi \to 0$ by virtue of the Gauss's theorem. Hence, it is shown that the function $g(\xi)$ does not even remain bounded at the origin. In addition, equation (19) can be made consistent if we impose the condition that the constant C_2 therein is zero. Making use of the formula $1/\Gamma(-n) = 0$ for $n = 0, 1, \ldots,$ we see that $C_2 = 0$ if and only if a or b is a non-positive integer. In such a case in which

$$ a = -n, \qquad b = n + 1 + \mu, \qquad C_2 = 0 \tag{24} $$

for each $n \in \mathbb{N}$, the constant C_1 reduces to

$$ C_1 = \frac{(c - b)_n}{(c)_n} = \frac{(\frac{1}{2} - n)_n}{(\mu + \frac{3}{2})_n} = \frac{(-1)^n(\frac{1}{2})_n}{(\mu + \frac{3}{2})_n} \tag{25} $$

and equation (19) becomes uniformly an identity over $\xi \in [0, 1]$. In particular, it is an easy matter to check that the two hypergeometric form of the desired solution have been successfully matched at $\xi = 0$ as well.

In conclusion, we have the occasion of stating our main result. The supplementary condition $a = -n$ in (24), in accordance with (16), may be interpreted as a *quantisation condition* from which we determine the even-parity state eigenvalues

$$ E(\mu) := E_{2n}(\mu) = \frac{1}{4}(2n + 1 + \mu)^2 \tag{26} $$

and the corresponding eigenfunctions

$$ \Psi_s(\xi; \mu) := \Psi_{2n}(\xi; \mu) = A_n(\mu)\,\xi^{\frac{1}{2}(\mu+1)}\, {}_2F_1(-n, n + 1 + \mu; \mu + \frac{3}{2}; \xi) \tag{27a} $$

$$ = A_n(\mu)\,\frac{(-1)^n(\frac{1}{2})_n}{(\mu + \frac{3}{2})_n}\,\xi^{\frac{1}{2}(\mu+1)}\, {}_2F_1 \times(-n, n + 1 + \mu; \frac{1}{2}; 1 - \xi) \tag{27b} $$

analytically for all $\mu \geqslant 0$, $0 \leqslant \xi \leqslant 1$ and $n \in \mathbb{N}$, where $A_n(\mu)$ is some normalisation constant. Returning back to the previous variables, we have

$$\Psi_{2n}(x;\mu) = A_n(\mu)\left(\frac{1}{2}+\frac{1}{2}x\right)^{\frac{1}{2}(\mu+1)} \times {}_2F_1\left(-n, n+1+\mu; \mu+\frac{3}{2}; \frac{1}{2}+\frac{1}{2}x\right) \tag{28a}$$

$$= A_n(\mu)\frac{(-1)^n(\frac{1}{2})_n}{(\mu+\frac{3}{2})_n}\left(\frac{1}{2}+\frac{1}{2}x\right)^{\frac{1}{2}(\mu+1)} \times {}_2F_1\left(-n, n+1+\mu; \frac{1}{2}; \frac{1}{2}-\frac{1}{2}x\right) \tag{28b}$$

for $x = \cos\theta \in [-1, 1]$, and

$$\Psi_{2n}(\theta;\mu) = A_n(\mu)\cos^{\mu+1}\frac{1}{2}\theta\, {}_2F_1\left(-n, n+1+\mu; \mu+\frac{3}{2}; \cos^2\frac{1}{2}\theta\right) \tag{29a}$$

$$= A_n(\mu)\frac{(-1)^n(\frac{1}{2})_n}{(\mu+\frac{3}{2})_n}\cos^{\mu+1}\frac{1}{2}\theta \times {}_2F_1\left(-n, n+1+\mu; \frac{1}{2}; \sin^2\frac{1}{2}\theta\right) \tag{29b}$$

for $\theta \in [-\pi, \pi]$. Note the relations

$$\xi = \frac{1}{2}(1+x) = \frac{1}{2}(1+\cos\theta) = \cos^2\frac{1}{2}\theta = u^2 \tag{30}$$

and

$$1-\xi = \frac{1}{2}(1-x) = \frac{1}{2}(1-\cos\theta) = \sin^2\frac{1}{2}\theta = t^2 \tag{31}$$

between the various variables. From now on, we assume that x is the main and the others are auxiliary variables.

3. The $\mathcal{T}$-polynomials of the first kind

The exact analytical solution for the symmetric states of the quantum mechanical problem has been asserted in section 2. It is clear that the hypergeometric factors in the eigenfunctions Ψ_{2n} stand for polynomials of degree n in the respective arguments. We introduce the name $\mathcal{T}$-polynomials of the first kind for these polynomials suggested by the trigonometric potential in (2), which are actually trigonometric in nature. Explicitly, if the normalising constant is specified to be

$$A_n(\mu) := (-1)^n \frac{(\mu+\frac{3}{2})_n}{(\mu+\frac{1}{2})_n} \mathcal{A}_n(\mu), \tag{32}$$

we rewrite the wavefunction (28a) and (28b) in the form

$$\Psi_{2n}(x;\mu) = \mathcal{A}_n(\mu) \left(\frac{1}{2}+\frac{1}{2}x\right)^{\frac{1}{2}(\mu+1)} \mathcal{T}_n^{(\mu)}(x) \tag{33}$$

in which the $\mathcal{T}$-polynomials of order μ and degree n in x have been denoted and defined by

$$\mathcal{T}_n^{(\mu)}(x) := \frac{(\frac{1}{2})_n}{(\mu+\frac{1}{2})_n}\, {}_2F_1\left(-n, n+1+\mu; \frac{1}{2}; \frac{1}{2}-\frac{1}{2}x\right) \tag{34a}$$

$$= (-1)^n \frac{(\mu+\frac{3}{2})_n}{(\mu+\frac{1}{2})_n}\, {}_2F_1\left(-n, n+1+\mu; \mu+\frac{3}{2}; \frac{1}{2}+\frac{1}{2}x\right), \tag{34b}$$

where $n \in \mathbb{N}$ and $\mu \geqslant 0$. Using the finite sum formula for the hypergeometric function in (34a), we have

$$\mathcal{T}_n^{(\mu)}(x) = \frac{(\frac{1}{2})_n}{(\mu+\frac{1}{2})_n} \sum_{k=0}^{n} \frac{(-n)_k (n+1+\mu)_k}{(\frac{1}{2})_k 2^k k!} (1-x)^k \tag{35}$$

from which the first three $\mathcal{T}$-polynomials

$$\begin{aligned} \mathcal{T}_0^{(\mu)}(x) &= 1, \\ \mathcal{T}_1^{(\mu)}(x) &= \frac{1}{2\mu+1}[(\mu+2)x-(\mu+1)], \\ \mathcal{T}_2^{(\mu)}(x) &= \frac{1}{(2\mu+1)(2\mu+3)}[(\mu+3)(\mu+4)x^2-2(\mu+1)(\mu+3)x+\mu^2+\mu-3] \end{aligned} \tag{36}$$

are listed at once. Furthermore, special values worth noting are

$$\mathcal{T}_n^{(\mu)}(1) = \frac{(\frac{1}{2})_n}{(\mu+\frac{1}{2})_n}, \qquad \mathcal{T}_n^{(\mu)}(-1) = (-1)^n \frac{(\mu+\frac{3}{2})_n}{(\mu+\frac{1}{2})_n} = (-1)^n \left(1+\frac{2n}{2\mu+1}\right), \tag{37}$$

which can be derived immediately from the definition (34). Unlike the ultraspherical polynomials, note that the $\mathcal{T}$-polynomials are neither even nor odd in which all powers of x are present.

Since the operator H in (2) is self-adjoint, the eigenfunctions of the original variable θ

$$\Psi_{2n}(\theta;\mu) = \mathcal{A}_n(\mu) \cos^{\mu+1}\frac{1}{2}\theta\; \mathcal{T}_n^{(\mu)}(\cos\theta) \qquad n \in \mathbb{N}, \tag{38}$$

form an orthogonal sequence over $\theta \in (-\pi, \pi)$. The elements of this sequence may be normalised if

$$\langle \Psi_{2m}, \Psi_{2n} \rangle = \mathcal{A}_m \mathcal{A}_n \int_{-\pi}^{\pi} \cos^{2\mu+2} \frac{1}{2}\theta \ \mathcal{T}_m^{(\mu)}(\cos\theta)\ \mathcal{T}_n^{(\mu)}(\cos\theta)\mathrm{d}\theta = \delta_{mn}, \tag{39}$$

which reads as

$$\mathcal{A}_m \mathcal{A}_n \int_{-1}^{1} (1-x)^{-\frac{1}{2}}(1+x)^{\mu+\frac{1}{2}} \mathcal{T}_m^{(\mu)}(x)\mathcal{T}_n^{(\mu)}(x)\mathrm{d}x = 2^{\mu}\delta_{mn} \tag{40}$$

with the change of variable $x = \cos\theta$ in (39). Consequently, the $\mathcal{T}$-polynomials are orthogonal on $(-1, 1)$ with a continuous and non-negative weight of the form

$$\rho(x) = (1-x)^{\alpha}(1+x)^{\beta}, \tag{41}$$

with $\alpha = -1/2$ and $\beta = \mu + 1/2$. It is known from the theory of special functions that the set of Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}$ for $\alpha, \beta > -1$ is the only orthogonal polynomial system over $(-1, 1)$ relative to the weight $\rho(x)$ [1]. More precisely, the $\mathcal{T}$-polynomials should be Jacobi polynomials for which $\alpha = -1/2$ and $\beta = \mu + 1/2$ apart from a suitable multiplicative constant, i.e. $\mathcal{T}_n^{(\mu)}(x) = d_n P_n^{(-\frac{1}{2},\mu+\frac{1}{2})}(x)$. Comparing the hypergeometric forms of the $\mathcal{T}$-polynomials with those of the Jacobi polynomials we can find out the rescaling constant d_n and establish the link

$$\mathcal{T}_n^{(\mu)}(x) = \frac{n!}{(\mu+\frac{1}{2})_n} P_n^{(-\frac{1}{2},\mu+\frac{1}{2})}(x). \tag{42}$$

The normalisation condition in (40) now gives

$$\mathcal{A}_n^2(\mu) = \frac{(\mu+\frac{1}{2})_n}{n!(\frac{1}{2})_n} \frac{\Gamma(\mu+1+n)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\mu)} \left(\frac{1+\mu+2n}{1+2\mu+2n}\right) \tag{43}$$

on employing the orthogonality property of $P_n^{(-\frac{1}{2},\mu+\frac{1}{2})}(x)$ [9].

It is worthwhile to emphasize that the $\mathcal{T}$-polynomials are not ultraspherical polynomials; however, they represent another subclass in the most general class of Jacobi polynomials. Recognition of this fact leads to an economical determination of the required properties of $\mathcal{T}_n^{(\mu)}(x)$. An alternative way is the use of the explicit form (35) and the differential equation (12). Actually, substituting the formula (26) for the eigenvalues E of the Schrödinger Hamiltonian into (12) we see that the $\mathcal{T}$-polynomials satisfy the equation

$$(1-x^2)y'' + [\mu+1-(\mu+2)x]y' + n(n+1+\mu)y = 0, \tag{44}$$

which may be written in the self-adjoint form

$$\left[(1-x^2)\rho(x)y'\right]' + n(n+1+\mu)\rho(x)y = 0, \tag{45}$$

where the weight $\rho(x) = (1-x)^{-1/2}(1+x)^{\mu+1/2}$ is a solution of the separable equation

$$\left[(1-x^2)\rho(x)\right]' = [\mu+1-(\mu+2)x]\rho(x) \tag{46}$$

of the first order. It follows then that the Rodrigues' formula for $\mathcal{T}_n^{(\mu)}(x)$ is

$$\mathcal{T}_n^{(\mu)}(x) = \frac{(-1)^n}{2^n(\mu+\frac{1}{2})_n}\frac{1}{\rho(x)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\rho(x)(1-x^2)^n\right], \tag{47}$$

which may also be verified directly, using the explicit formula (35). Similarly, among the others, a recurrence relation of the form

$$\mathcal{T}_{n+1}^{(\mu)}(x) = [\,2a_n(\mu)x - b_n(\mu)\,]\mathcal{T}_n^{(\mu)}(x) - c_n(\mu)\mathcal{T}_{n-1}^{(\mu)}(x) \tag{48}$$

may be derived, in which the coefficients $a_n(\mu)$, $b_n(\mu)$ and $c_n(\mu)$ are

$$a_n(\mu) = \frac{(2n+\mu+1)(2n+\mu+2)}{(2n+2\mu+1)(2n+2\mu+2)}, \tag{49a}$$

$$b_n(\mu) = \frac{\mu(\mu+1)(2n+\mu+1)}{(2n+\mu)(2n+\mu+1)(2n+2\mu+1)}, \tag{49b}$$

and

$$c_n(\mu) = \frac{2n(2n-1)(2n+\mu+2)}{(2n+\mu)(2n+2\mu-1)(2n+2\mu+2)}, \tag{49c}$$

respectively.

4. The limiting case $\mu \to 0$ and concluding remarks

We have investigated the exactly solvable quantum mechanical oscillator with the trigonometric potential

$$v(\theta) = \frac{1}{4}\mu(\mu+1)\sec^2\frac{1}{2}\theta = \frac{1}{2}\mu(\mu+1)(1+\cos\theta)^{-1}, \qquad \mu \geqslant 0 \tag{50}$$

having an enumerable infinite set of discrete positive spectral points, which corresponds to a complete sequence of orthonormal eigenfunctions over $\theta \in (-\pi, \pi)$. Recall that most of the popular exactly solvable systems such as the Morse potential, have only finitely many eigenvalues [4, 10]. The spectral points given by the formula (26) are well separated and the simplicity of eigenvalue differences is quite similar to that of the classical harmonic oscillator.

It should be pointed out that any exactly solvable Hamiltonian is interesting from a different point of view. For, such a system serves as a simple model for understanding the more realistic and complex problems. Actually, the most popular harmonic oscillator models have played a tremendous role in the evolution of many fields of physics [11]. Furthermore, they may provide a basis for pseudo-spectral, spectral and perturbation methods for differential eigenvalue problems. To be specific, a more general problem with a potential function of the form

$$q(\theta) = \frac{1}{4}\mu(\mu+1)\sec^2\frac{1}{2}\theta + \varepsilon w(\theta), \qquad \varepsilon \in \mathbb{R}, \tag{51}$$

is most easily attacked by considering $w(\theta)$ as a perturbation of our trigonometric potential, where ε is a parameter. In a perturbation series, it is generally the existence of a closed form zeroth-order solution for $\varepsilon = 0$ which ensures that the higher-order terms may also be determined as closed form analytical expressions [12].

The limit particle-in-a-box potential as $\mu \to 0$ is of particular importance. In this case, the normalisation constant (43) is simplified to $\mathcal{A}_n^2(0) = 1/\pi$, and the eigenfunctions (38) becomes

$$\Psi_{2n}(\theta; 0) = \frac{1}{\sqrt{\pi}} \cos\tfrac{1}{2}\theta\, \mathcal{T}_n^{(0)}(\cos\theta). \tag{52}$$

Comparing this with the exact solution (26) of the limit potential we infer that there must be the interrelation

$$\mathcal{T}_n^{(0)}(x) = V_n(x) \tag{53}$$

between the $\mathcal{T}$-polynomials of the first kind of the zeroth-order and the Chebyshev polynomials of the third kind. From (36) and (8) we have $\mathcal{T}_0^{(0)}(x) = 1 = V_0(x)$ and $\mathcal{T}_1^{(0)}(x) = 2x - 1 = V_1(x)$. Since the coefficients are $a_n(0) = c_n(0) = 1$ and $b_n(0) = 0$ at $\mu = 0$, the recurrence relation (48) of the $\mathcal{T}$-polynomials turns out to be

$$\mathcal{T}_{n+1}^{(0)}(x) = 2x\mathcal{T}_n^{(0)}(x) - \mathcal{T}_{n-1}^{(0)}(x), \tag{54}$$

which is nothing but the recursion for the Chebyshev polynomials implying the truth of the interrelation (53) for all n. Moreover, both $V_n(x)$ and $\mathcal{T}_n^{(\mu)}(x)$ are neither even nor odd, and the differential equation (44) for the $\mathcal{T}$-polynomials reduces to

$$(1 - x^2)y'' + (1 - 2x)y' + n(n+1)y = 0 \tag{55}$$

for $\mu = 0$, which is the equation satisfied by $V_n(x)$ [5]. Thus the $\mathcal{T}$-polynomials of the first kind can be viewed as a generalisation of the Chebyshev polynomials of the third kind. It is an easy matter to find the connection between $\mathcal{T}_n^{(\mu)}$ and the Chebyshev polynomials T_n of the first kind as well. In fact, using the definitions of V_n and T_n and the link in (53) we see that

$$\mathcal{T}_n^{(0)}(x) = \frac{1}{u} T_{2n+1}(u), \tag{56}$$

where u is one of the auxiliary variables in (30).

As a final remark, the treatment of the antisymmetric states of the present trigonometric oscillator along the same lines is in progress. There is enough evidence to claim that it will be possible to describe another class of polynomials in this context, which may be called accordingly the $\mathcal{T}$-polynomials of the second kind.

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